WHERE MATHEMATICS COMES FROM

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Part I

Chapter 1

.The Embodiment of Basic Arithmetic

:The goal of this book is to answer the questions

?What is the cognitive structure of sophisticated mathematical ideas -

What are the simplest mathematical ideas, and how we build on them and extend them to develop complex mathematical ideas: the laws of arithmetic, set theory, ?logic complex numbers, limits, and so on -

The authors begin with the most fundamental aspect of numbers and arithmetic, the .part that we are all born with

Number Discrimination by Babies 1.1

Recent research has shown that babies have the following numerical abilities: (this (."abilities call "**innate arithmetic**

At three or four days, a baby can discriminate between collections of two and three items. Under certain conditions, infants can even distinguish three items from .four -

By four and a half months, a baby "can tell" that one plus one is two and that two .minus one is one -

A little later, infants "can tell" that two plus one is three and that three minus one .is two -

At three or four days, a baby can discriminate between sounds of two or three .syllables -

And about seven months, babies can recognize the numerical equivalence .between arrays of objects and drumbeats of the same number -

.They know that babies can make these numerical distinctions by experimental

Subitizing 1.2

Definition: Subitize is the ability to tell at a glance whether there are one, two, or .three objects before them _

.It is the ability that allows new born babies to make the distinctions discussed above

To subitizing beyond 4 we need extra time and extra cognitive operations-grouping .the objects into smaller, subitizable groups and counting them

.A comment: subitizing is a different process from counting or estimating

The ability to subitize is inborn. The classing subitizing experiment involves reaction time and accuracy. A number of items are flashed before subjects for a fraction of .second and they have to report as fast as they can how many there are

The Numerical Abilities of Animals 1.3

Animals have numerical abilities, not just primates but also raccoons, rats, parrots and pigeons. They can subitize, estimate numbers, and do the simplest addition and .subtraction, just as four-and-a-half-month-old babies can

From Brain to Mind and from Basic Arithmetic to Mathematics 1.4

They have some idea of what areas of the brain are active when use such capacities, and they have some idea of which if these capacities are innate. But knowing **where** is far from knowing **how.** To know what parts of the brain "light up" when certain task are performed is far from knowing the neural mechanism by which those tasks .are performed

The hard question is how to go from such simple capacities to sophisticated forms of .mathematics

Citation of the authors: "We are born with a minimal innate arithmetic, part of which :we share with others animals. Innate arithmetic includes at least two capacities

a capacity for subtizing-instantly recognizing small numbers of items, and (1)
".a capacity for simplest forms of adding and subtracting small numbers (2)

Chapter 2

A Brief Introduction to the Cognitive Science of the Embodied Mind

The Cognitive science of mathematics is a new discipline, not much is known for sure right now about just how mathematical cognition works. The aim of the book is to explore how the general cognitive mechanisms used in everyday non-mathematical .thought can create mathematical understanding and structure mathematical ideas

Ordinary Cognition and Mathematical Cognition 2.0

As we saw, it appears that all human being are born with capacity for subitizing very small numbers. But there is a lot more to mathematics that the arithmetic of very small numbers, such as trigonometry calculus and so on. Extending numbers to the rational, the real, the imaginaries requires an enormous cognitive apparatus and goes

well beyond what babies and animals, and even normal adult without instruction, can .do

The remainder of the book will be concerned with the **embodied** cognitive capacities that allow one to go from innate basic numerical abilities to a deep and rich .understanding of college level mathematics

This chapter presents prominent examples of the kinds of everyday conceptual mechanisms that are central in human being. The mechanisms are (a) image schemas, .(b) aspectual schemas, (c) conceptual metaphor, and (d) conceptual blends

.Spatial Relations Concepts and Image Schemas 2.1

Every language has a system of spatial relation, though they differ radically from .language to language

Research in cognitive linguistics has shown that spatial relations in a given language decompose into conceptual primitive called "*image schemas*" and it appears to be .universal

For example, the English word *on*, in the sense used in "The book is *on* the desk," is a :composite of three primitive image schemas

(The *Above schema* (the book is *above* the desk :1

(The *Contact schema* (the book is in *contact* with the desk :2

(The *Support schema* (the book is *support* by the desk :3

The above schema is orientational, it specifies an orientation in space relation to the - .gravitational pull one feels on one's body

The contact schema is one of a number of topological schemas, it indicates the - .absence of a gap

The support schema is force-dynamic in nature, it indicate the direction and nature of- .a force

Not all language have a single concept like the English *on*. In German, the *on* in "*on* .the table" is rendered as *auf*, while the *on* in "*on* the wall" is translated as *an*

A common image schema in mathematics is the *container schema* which has three .parts: (a) Interior (b) Boundary (c) Exterior

This structure forms a gestalt, in the sense that parts make no sense without the whole. There is no Interior without a Boundary and an Exterior, no Exterior without a .Boundary and an Interior, and so on

.Image schema is the link between language and spatial perception

There are many image schemas that characterize concept important for mathematics: centrality, contact, closeness, balance, straightness, and many more. Image schemas .and their logics are essential to mathematical reasoning

Motor Control and Mathematical Ideas 2.2

(.Motor control is the neural system that governs how we move our bodies)

Recent discoveries suggest that our neural motor-control system may be centrally .involved in mathematical though
David Bailey and Srini Narayanan's model (1997) has observed that neural motor-:control programs all have the same superstructure

Readiness; Before you can perform a bodily action, certain conditions of (1) readiness have to be met (e.g., you may have to reorient your body, stop doing .something else, rest for a moment, and so on
.Starting up (2)
.The main process (3)
.Possible interruption and resumption (4)
.Iteration or continuing (5)
.Purpose (6)
.Completion (7)
.Final state (8)
(.The explanation is in pages 34-35)

These are all necessary properties for the smooth function of neural motor-control system. Everything that we perceive or think of as an action or event is .conceptualized as having that structure

Narayanan's model tells us that "the same neural structure used in the control of ".complex motor schemas can also be used to reason about events and action
.This is the *Aspect schema*

Among the logical entailment of the aspectual system are two inferential patterns :important for mathematics

The stage characterizing the completion of the process is further along relative to - .the process than any stage within the process itself
.There is no point a process further along than the completion stage of that process-

.These fairly obvious inferences, as we shall see in chapter 8 on infinity

?**Question**. What is the connection between the aspect schema and math ideas

The Source-Path-Goal Schema 2.3

Every language includes ways of expressing spatial source (e.g., "from") and goals (e.g., "to," "toward") and paths intermediate between them (e.g., "along," "through," .(""across

The Source-Path-Goal Schema
is the principal image schema concerned with motion, and it has the following :(elements (or roles
.A trajectory that moves (1)
.A source location (2)
.A goal (3)
.A rout from the source to the goal (4)
.The actual trajectory of motion (5)

.The position of the trajectory at a given time (6)
.The direction of the trajectory at that time (7)
.The actual final location of the trajectory (8)

The Source-Path-Goal Schema is ubiquitous in mathematical thought. For example .functions in the Cartesian plan are conceptualized in terms of motion along a path

Conceptual Metaphor 2.4

Metaphor, long thought to be just a figure of speech, has recently been shown to be a central process in everyday thought. **Metaphor** isn't a mere embellishment, **it's the basic means by which abstract thought is made possible**. One of the principle results in cognitive science is that abstract concepts are typically understood, via .metaphor, in terms of more concrete concept

For example, affection, is understood in terms of physical warmth, as in sentences like
","she *warmed* up to me
",You've been *cold* to me all day"
",He gave me *icy* stars"
".They haven't yet broken the *ice*"

The metaphor is not a matter of words, but conceptual structure. The words are all different (warm, cold, icy, ice), but the conceptual relationship is the same in all cases: Affection is conceptualized in terms of *warmth* and Disaffection in terms of *.cold*
(For more read see (Lakoff & Johnson, 1980, 1999; Grady, 1998; Nunez, 1999

Metaphor is used unconsciously, effortless, and automatically in everyday discourse, .that is, they are part of the cognitive unconscious

Metaphor is a mapping from entities in one conceptual domain to corresponding entities in another conceptual domain, i.e. from source-domain concept to target-domain concept. They write it in the form A→B, i.e., the source domain is to the left .arrow and the target domain is to the right

?**Q**. What is the meaning of the tables in pages 42-44

Metaphors That Introduce Elements 2.5

.Conceptual metaphors can also introduce new elements into the target domain
:For example

Source Domain Business		Target Domain Love
Partners	→	Lovers
Partnership	→	Love relationship
Wealth	→	Well-being

Love need not always conceptualized via this metaphor as a partnership but this is a .common metaphorical way of understanding love
We shall see later that metaphor that introduces elements into a target domain is .important for mathematics

Sophisticated Mathematical Ideas 2.6

The argument of the authors is that *conceptual metaphor is the central cognitive mechanism of extension from basic arithmetic to such sophisticated applications of numbers*. Moreover they argue that sophisticated understanding of arithmetic itself requires conceptual metaphors using non-numerical mathematical source domain .((e.g., geometry and set theory
Finally, much of the "abstract" of higher mathematics is a consequence of the systematic layering of metaphor upon metaphor. Each metaphorical layer carries inferential structure systematically from source domain to target domain-systematic structure that gets lost in the layers unless they revealed by detailed metaphorical .analysis

Symbolization 2.6.1

In embodied mathematics, mathematical symbols, like 27, π,… are meaningful by virtue of the mathematical concepts that they attach to. Numerical calculation may be performed with or without genuine understanding. The meaning of mathematical symbols is not in the symbols alone and how they can be manipulated by rule. Mathematical meaning is like everyday meaning. It is pare of embodied cognition. This is important consequences for the teaching of mathematics. Rote learning and drill is not enough. It leaves out understanding. Similarly, deriving theorems from formal axioms via purely formal rules of proof is not enough. It, too, can leave out understanding. The point is not to be able to prove that e^(πi)=-1 but, rather, to be able to prove it knowing what e^(πi) means, and knowing why e^(πi)=-1 on the .basis of what e^(πi) means, not just on the basis of the formal proof

Conceptual Blends 2.6.2

A Conceptual blend is the conceptual of two distinct cognitive structures with fixed .correspondences between them

.**Q** The example of the circle isn't clear

When the fixed correspondences in a conceptual blends are given by a metaphor, they .call it a *metaphorical blend*

Chapter 3
Embodied Arithmetic: The Grounding Metaphors

Arithmetic is a lot more than subitizing and the elementary numerical capacities of monkeys and newborn babies. To understand what arithmetic is from a cognitive :perspective, we need to know much more

?Why does arithmetic have the properties it has-
?Where do the laws of arithmetic come from-
What cognitive mechanisms are needed to go from what we are born with to full--
?blown arithmetic

?What is Special about Mathematics 3.0

:Arithmetic in particular and mathematics in general are special in several ways

,precise-
,consistent-
,stable across time and communities-
,understandable across cultures-
,symbolizable-
,calculable-
generalizable, and-
effective as general tools for description, explanation, and prediction in a vast -
number of everyday activities, from business to building to sports to science and
.technology

Any cognitive theory of mathematics must take these special properties into account, showing how they are possible given ordinary human cognitive capacities. That the .goal of this chapter

The Cognitive Capacities Needed for Arithmetic 3.1

We are born with a minimal innate arithmetic. Innate arithmetic includes at least two capacities: (1) a capacity for subitizing-instantly recognizing small numbers if items
and
a capacity for simplest forms of adding and subtraction small numbers. (2)
("number=" cardinal number= a number that specifies how many objects there are in
(.a collection

Arithmetic involves more than a capacity to subitize and estimate. Subitizing is certain and precise within its range. But we have additional capacities that allow us to extend this certainty and precision. To do this we must count , so

:the **cognitive** capacities needed in order to count is

Grouping capacity: To distinguish what we are counting, we have to be able to -
.group discrete elements visually, mentally, or by touch
.Ordering capacity: the items have to place in a sequence -
.Pairing capacity -
.Memory capacity -

Exhaustion-detection capacity: the ability to tell when there are "no more" -
.objects left to be account
Cardinal-number assignment: the able to assign the last number in the count as -
.the size of the group
Independent-order capacity: the size of the group is independent of the order of -
.the group

When these capacities are used within the subtizing range between 1-4 we get stable :results. To count beyond 4 we need additional capacities

Combinatorial-grouping capacities: we need a cognitive mechanism that allows -
.us to put together perceived or imagined groups to form larger groups
Symbolizing capacities: the able to associate physical symbols or words with -
.numbers

But subitizing and counting are the bare of arithmetic. To go beyond them, to characterize arithmetic operations and their properties, we need much richer cognitive :capacities

Metaphorizing capacity: the able to conceptualizing cardinal numbers and -
arithmetic operations in terms of our experience of various kinds-experiences with .groups of objects, with distances, with movement and location, and so on
Conceptual-blending capacities: the able to combining subitizing with counting, -
and to be able to put together different conceptual metaphors to form complex .metaphors

A comment: The metaphorizing capacity is central to the extension of arithmetic .beyond mere subitizing, counting and simplest adding and subtraction

?**Q**. The "Metaphorizing capacity" is not clear

תשובה אפשרית: כנראה שזו היכולת להבין או לתפוס את המושג מספר ופעולות אריתמטיות במושגים של סוגי החוויות השונות שלנו.

These two metaphors are the most basic cognitive mechanisms that take us beyond innate arithmetic and simple counting to the elementary arithmetic of .natural numbers

They (the authors) found two types of **conceptual metaphor** used in projective from subitizing, counting, and simplest arithmetic of newborns to an arithmetic of **natural** :numbers

The first is the **grounding metaphors**: metaphors that allow us to project -
from everyday experience (like putting things into piles) onto abstract concept (like addition) In other words these metaphors are ground our understanding of mathematical ideas in terms of everyday experience. It grounds our conception of arithmetic directly in an everyday activity, for example: subtraction as taking objects .away from a collection
חוויות יומיומיותeveryday experiences=

The second is the **linking metaphors**: which link arithmetic to other branches - .of mathematics, for example: geometrical figures as algebraic equations

A comment : The grounding metaphors and the linking metaphors is two types of .conceptual metaphor

This chapter is devoted to grounding metaphor. The rest of the book is devoted .primarily to linking metaphors

Such metaphor allows us to ground our understanding of arithmetic in our prior understanding of extremely commonplace physical activities. Our understanding of :elementary arithmetic is based on a correlation between

The most basic literal aspects of arithmetic, such as subitizing and counting (1)
and
Everyday activities, such as collecting objects into groups, taking objects apart (2) .and putting them together, and so on

Arithmetic As Object Collection 3.2

Arithmetic Is Object Collection is grounding metaphor, because it grounds our conception of arithmetic directly in an everyday activity. No metaphor is more basic to the extension of our concept of number from the innate cardinal numbers to the natural numbers. The reason is that the correlation of grouping with subitizing and counting the elements in a group is pervasive in our experience from earliest .childhood

The metaphor Arithmetic Is Object Collection is learned at an early age, prior to any formal arithmetic training. In teaching arithmetic, we all take it for granted that the adding and subtraction of numbers can be understood in terms of adding and taking away object from collection. All of these stages are mental operations *with no .symbols!* Calculating with symbols requires additional capacities

The Arithmetic Is Object Collection metaphor is a precise ***mapping*** from the domain of physical objects to the domain of numbers (it's limited to addition and subtraction). Operations in one domain (using only collections) are mapped onto operation in the .(other domain (using only numbers

:The metaphorical mapping consists of

,The source domain of object collection (1)
,The target domain of arithmetic (2)
.A mapping across the domains (3)

Arithmetic Is Object Collection

Source domain Object Collection	Target domain Arithmetic
→Collections of objects of the same size	Numbers
The size of the collection →	The size of the Number

→ The smallest collection	(The unit (One
Putting collection together →	Addition

.And so on

Linguistic Example of the Metaphor 3.2.1

We can see evidence of this conceptual metaphor in our everyday language. The word *add* has the physical meaning of physically placing substance or a number objects into a container (or group of objects), as in "*Add* sugar to my coffee"… In the same way for subtraction, *take…from, take…out*, *take…away* have the physical meaning of removing a substance, an object, or a number of objects from some container or .collection

Q If we can connect the "container" here to the container schema that appears in page ?31

There is a table in the book that shows that this metaphor extends the innate 3.2.3 subitized numbers 1 through 4 to an indefinitely large collection of natural numbers. We can see clearly how properties of Object Collection are mapped by the metaphor :onto properties of natural numbers in general. For example

Magnitude

→ *Object collection* have a magnitude	*Numbers* have magnitude

Inverse Operations

For *collection:* Whenever we subtract what we added, or add what we .subtracted, we get the original *collection*	For *Numbers:* Whenever we subtract what we added, or add what we .subtracted, we get the original *number*

.And so on, in pages 57-59

In summery, this metaphor will extend innate arithmetic, adding properties that the innate arithmetic of numbers 1 through 4 does not have, because of its limited range. The metaphor will map these properties from the domain of object collections to the expanded domain of number. The result is the elementary arithmetic of addition and .subtraction for natural numbers, *which goes beyond innate arithmetic*
Thus, the fact that there is an innate basis for arithmetic does not mean that all .arithmetic is innate

Extending Elementary Arithmetic 3.3

The Arithmetic As Object Collection metaphor limited to addition and subtraction operation. The cognitive mechanism that allows us to extend this metaphor from addition and subtraction to multiplication and division is *metaphrtic blending* i.e. simultaneous activation of two domains with connections across the domains. Again, with multiplication, we do need to refer to numbers and collections simultaneous, since understanding multiplication in terms of collections requires performing

operations on collections a certain numbers of times. This cannot be done in a domain .with collections alone or numbers alone

Two Version of Multiplication and division 3.3.1

.The two version of multiplication is (1) by *pooling*, and (2) by *repeated addition*
In the same way we can describe division in two corresponding ways: (1) *splitting up*, .and (2) *repeated subtraction*

.The example is in page 61

In each of these cases we have used numbers with only addition and subtraction defined in order to characterize multiplication and division metaphorically in terms of object collection. From a cognitive perspective, we have used a metaphoric blend of object collections together with numbers to extend the Arithmetic Is Object .Collection metaphor to multiplication and division

In order to characterize multiplication and division metaphorically in terms of object .collection we can use numbers with only addition and subtraction

The *Pooling/Splitting* Extension of the Arithmetic is Object Collection Metaphor

Source domain The Object-Collection/Arithmetic blend	Target domain Arithmetic
The *pooling* of **A** sub-collections of size **B** to form an overall collection of size **C** →	(Multiplication (**A*B=C**
The *splitting* up of a collection of size **C** → into **A** sub-collections of size **B**	(Division (**C÷B=A**

The *Iteration* Extension of the Arithmetic is Object Collection Metaphor

Source domain The Object-Collection/Arithmetic blend	Target domain Arithmetic
The *repeated* addition (**A** times**)** of a collection of size **B** to yield a collection → of size **C**	(Multiplication (**A*B=C**
The *repeated* sub-traction of collection of size **B** from an initial collection of size **C** until the initial collection is exhausted. **A** is the number of times the sub-traction → occurs	(Division (**C÷B=A**

.And it's easy to define the commutative, associative, and distributive properties
Thus, the Arithmetic Is Object Collection metaphor extends our understanding of number from the subitized numbers of innate arithmetic and from simple counting to the arithmetic of the natural numbers, grounding the extension of arithmetic in our .everyday experience with groups of physical objects

Zero 3.3.2

"A new conceptual metaphor is necessary to define "*not a collection of objects*

The Zero Collection Metaphor

The lack of objects to form a → collection	The empty collection

The Arithmetic As Object Collection metaphor will map the empty collection onto a .number which they call "*Zero*." This metaphor calls an *entity-creating metaphor* This metaphor does not arise from a correlation between the experience of collecting and the experience of subitizing and doing innate arithmetic. It is therefore an .artificial metaphor
.And it easy to define the additive identity and the inverse of addition laws

These metaphors ground our most basic extension of arithmetic-from innate cardinal .numbers to the natural numbers plus zero

.Until now we have one metaphor. There are three more to go

Arithmetic As Object Construction 3.4

How is it possible to understand a number, which is an abstraction, as being "made up," or "composed of," other number, which we are "put together" using arithmetic operations? What they are doing here is conceptualizing numbers as wholes made up of parts. The parts are other numbers. And the operations of arithmetic provide the .patterns by which the parts fit together to form wholes
:Here is the metaphorical mapping used to conceptualize numbers in this way

Arithmetic Is Object Construction

Source domain Object construction	Target domain Arithmetic
Objects →	Numbers
The smallest whole object →	(The unit (One
→ The size of the object	The size of the Number
Acts of object construction →	Arithmetic operations
A constructed object →	The result of an arithmetic operation
Putting object together with other to form → larger objects	Addition
Taking smaller objects from larger → objects to form other objects	Subtraction

As in the case of Arithmetic As Object Collection, this metaphor can be extended in two ways via metaphorical blending*: fitting together/splitting up* and *iterated .addition and subtraction*

Q. What is the different between the metaphor above (Arithmetic As Object ?(Construction and Arithmetic As Object Collection
Maybe it's another way to understand numbers and operation between the numbers.) ((?(the answer is in page 67

As in the case of the object-collection metaphor, a special additional metaphor is :needed to conceptualize zero

The Zero Object Metaphor

The lack of whole objects →	Zero

Object construction metaphor is more specific version of object collection metaphor. It also has metaphorical entailments that characterize the decomposition of numbers :into parts

Whole objects are composites of their parts, put together by → .certain operations	Whole numbers are composites of their .parts, put together by certain operations

The Measuring Stick Metaphor 3.5

We use a measuring stick or string taken as a unit. These are physical versions of what in geometry are called *line segment* they will refer to them as "physical segments." The physical segments are body parts: fingers, hands forearm, arms, feet and so on. A distance can be measured by placing physical segments of unit length end-to-end and counting them. When we put physical segments end-to-end, the result .is another physical segments

The Measuring Stick Metaphor

Source domain The use of a Measuring Stick	Target domain Arithmetic
Physical segments →	Numbers
The physical segment →	One
→ The length of the physical segment	The size of the Number
Acts of physical segment placement →	Arithmetic operations
A physical segment →	The result of an arithmetic operation
Putting physical segments together end-to-end with other physical segments to → form longer physical segments	Addition
Taking shorter physical segments from larger physical segments to form other → physical segments	Subtraction

And as before it's easy to characterizing the other operation as multiplication with .physical segments

A comment: This metaphor needs to be extended in order to get conceptualization of :zero

The lack of any physical → segment	Zero

Physical segments are continuous. There is a one-to-one correspondence between physical segments and numbers i.e. given a fixed unit length then for every physical .segment there is a number
The measuring stick metaphor allow one to form physical segments of particular numerical length such as √2, √3, √4, √5, √6, √7, √8, √9, √10, …It was the measuring stick metaphor and the Number/Physical Segment that gave birth to the irrational (.numbers. Exdoxus (c. 370 B.C

Arithmetic As Motion Along a Path 3.6

When we move in a straight line from one place to another, the path of our motion form a physical segment-an imagined line tracing our trajectory. There is a simple relationship between a path of motion and a physical segment. The origin of the motion corresponds to one end of a physical segment, the endpoint of the motion corresponds to the other end of the physical segment, and the path of the motion .corresponds to the rest of the physical segment

A comment: Pay attention between the above metaphor and the schema Source-Path-.Goal

Q The unit of the physical segment is a straight line, and the path can be curve path, how then we correspond between the unit physical segment and the curve path? .((maybe it's just metaphora

Given this correspondence between motion and physical segments, there is a natural metaphorical correlate to the measuring stick metaphor for arithmetic, namely, the :metaphor that Arithmetic Is Motion Along a Path

Arithmetic Is Motion Along a Path

Source Domain Motion Along a Path	Target Domain Arithmetic
→ Acts of moving along the path	Arithmetic operations
→ A point-location on the path	The result of an arithmetic operation
→ The origin, the beginning of the path	Zero
→ Point-locations on a path	Numbers
The unit location, a point-location → distinct from the origin	One
Moving from a point-location **A** away from the origin, a distance that is the same as the distance from the origin to a → point-location **B**	Addition of **B** to **A**
Moving toward the origin form **A,** from a distance that is the same as the distance → from the origin to **B**	Subtraction of **B** from **A**

.This metaphor can be extended to multiplication and division

This metaphor provides a natural extension to **negative numbers**: let fixed the origin somewhere on a pathway then we get both directions. The negative numbers will be

the point-location on the other side of zero from the positive numbers along the same path. This extension was explicitly made by Rafael Bombelli in the second half of the sixteenth century. In Bombelli's extension of the point-location metaphor for numbers, positive numbers, zero, and negative numbers are all point-location on a line. This made the concept of a numbers *lying between* two other numbers-as in *zero .lies between -1 and 1*
Conceptualizing all (real) numbers metaphorically as point-location on the same line .was crucial to providing a uniform understanding of numbers
The understanding of numbers as point-location has come into our language in the :following expressions

.How ***close*** are these two numbers? 37 are ***far away from*** 189,712-
.is ***near*** 5 4.9-
The result is ***around*** 40-
.Count up ***to*** 20 =, without ***skipping*** any numbers-
.Count ***backward*** from 20-
.Count ***to*** 100, start ***at*** 20-
.Name all the numbers ***from*** 2 ***to*** 10-

The linguistic examples illustrate how the language of motion can be recruited in a .systematic way to talk about arithmetic

Now the authors completed the description of the 4 basic grounding metaphors (i.e. *Object Collection*, *Object Construction*, *The Measuring Stick Metaphor, and Motion .Along a Path*) for arithmetic

The Metaphorical Meaning of One and Zero 3.7

The 4 grounding metaphors mentioned so far-Object Collection construction, the Measuring Stick, and Motion Along a line-contain metaphorical characterizations of .zero and one. These metaphors characterize the symbolic meaning of zero and one
In collection metaphor, zero is the empty collection in the object construction metaphor, zero is either the lack of an object, the absence of an object or, as a result …of an operation, the destruction of an object. Thus zero can mean *lack, absence*
In the measuring stick metaphor, zero is the lack of any physical segment. In the .motion metaphor, zero is the origin of motion, hence, zero can designate an *origin*
Hence, zero, in everyday language, can symbolically denote emptiness, nothingness, .lack, absence destruction, and origin
In the collection metaphor, one is the collection with a lone member and, hence, symbolizes *individuality* and *separateness* from other. In the object construction metaphor, one is a whole number and, by virtue of this, signifies *wholeness*, *unity*, and *integrity*. In the measuring stick metaphor, one is the length specifying the unit of measure and it signifies a *standard*. In the motion metaphor, one indicates the first .step in a movement, hence, it symbolizes a *beginning*
Taken together, these metaphors give one the symbolic value of individuality, separateness, wholeness, unity, integrity, a standard, and a beginning. Here are some :examples

.Beginning: One small step for a man, one great step for mankind-
Origin: Let's start again from zero-

.*Emptiness*: There's zero in the refrigerator-
.And so on

But the real importance for mathematics is that they explain how innate arithmetic gets extended systematically to give arithmetic distinctive properties that innate .arithmetic does not have

!**Q** The above sentence needed an explanation or interpretation

.The 4 grounding metaphor for arithmetic called: **4Gs**

Chapter 4
?Where Do the Laws of Arithmetic Come From

The significance of the 4Gs 4.1

Innate arithmetic, as we saw, is extremely limited: It includes only subitizing, addition, and subtraction up to the number 4 at most. The **4Gs** each arise via a conflation in everyday experience. Taken object collection, for example, young children form small collections, subitize them, and add and take away objects from them, automatically forming additions and subtractions within the subitizable range. These correlations in everyday experience between **innate arithmetic** and the **source** .**domain** of the 4Gs give rise to the 4Gs

The significance of the 4Gs is that they allow human being, who have an innate capacity to form metaphors, to extend arithmetic beyond the small amount that we are born with, while preserving the basic .properties of innate arithmetic

?**Q.** Is it a child that doesn't have an innate capacity to form the metaphor

:There is structural correspondence between

.object collection and object construction-
the construction of a linear object and the use of a measuring stick to mark off a - .line segment of certain length
using a measuring stick to mark off a line segment, or "path," and moving from - .location to location along a path

The correspondence is, For example, object construction always involves object .collection; you can't build an object without gathering the parts together
Putting physical segments end-to-end is similar to object construction (think of legos .here) and so on

As a result of this structural correspondence, there is isomorphism across the **4Gs** metaphor. For there to be such an isomorphism, the following three conditions must :hold

There is a one-to-one mapping, M, between elements in one source-domain and -
.elements in the other source-domain
(M(x+y) = M(x) + M(y -
(M(x*y) = M(x) * M(y -

.**A comment**: the isomorphism is between the elements of the source domain
For example: The source domain of object collection and motion along a path. First there is a one-to-one correspondence, M, between sizes of collections and distances moved. A collection of size 3 is uniquely mapped to a movement of 3 units of length, .and conversely. And it's easy to show that the conditions for isomorphism are valid

The source domains of all four basic grounding metaphors for the arithmetic of !**natural numbers** are isomorphic in this way

A comments: 1- There are no numbers in these source domains, there are only object .collections, motions, and so on
Object construction characterizes fractions but not zero or negative -2
numbers, whereas motion along a path characterizes zero or negative numbers, for example. In other words, if we look at the complete domains in isolation, we will not see an isomorphism across the source domains. **What creates the isomorphism is the collection of mappings from these source domain of the 4Gs onto natural numbers and what grounds the mappings onto natural numbers are the experience we have, across the four domains, with innate arithmetic-with subitizing and counting in such early experiences as forming collections, putting .things together, moving from place to place, and so on**

Number are Things 4.2

In each of the 4Gs, numbers are things that exist in the world: Collection of objects, complex objects with parts, physical segments, and location. These four metaphors .thus induce a more general metaphor: **Number Are Thing in the World**
Though they can function as quantifiers in everyday language (e.g., "five apples"), numbers in arithmetic statements function like things. The name of numbers (e.g., ."five") are proper nouns

4.3(סגירותClosure (

In most of our everyday experience, when we operate on actual physical entities, the result is another physical entity. If we put two objects together, we get another object. If we combine two collections, we get another collection, and so on. Thus the general :principle holds

.An operation on physical things yields a physical thing of the same kind-

:The metaphor that Number are Things yields a corresponding principle

An operation on numbers yields a number of the same kind-

.**Q** I disagree with the last conclusion for example: $\sqrt{2}*\sqrt{2}=2$

.The name for this metaphorical principle in mathematics is ***closure***

A comment: Closure is not a property of innate arithmetic. "Subitizing 3" plus ."subitizing 4" does not produce a subitizable number, we don't normally subitize 7

Closure is a central idea in mathematics. It has led mathematicians to extend number systems further and further until closure is achieved-and to stop with closure. Thus, the natural numbers had to be extended in various ways to achieve closure relative to the basic arithmetic operations (addition, subtraction, multiplication, division, rasing .(to power, and taking roots

:Examples

.Because 5-5 is not a natural number, zero had to be added-
.Because 3-5 is not a natural number, negative numbers had to be added-
.Because 3/5 is not a natural number, rational numbers had to be added-
Because $\sqrt{2}$ is not a rational number, irrational numbers had to be added to form -
."the "real numbers
Because $\sqrt{-1}$ is not a real number, the "imaginary numbers" had to be added to form -
".the "complex numbers

Extending the natural numbers to the complex numbers finally achieved closure relative to the basic operations of arithmetic. As the fundamental theorem of algebra .implies any arithmetic operation on any complex numbers yield a complex number

In this way, we can achieve a closure if given any set of mathematical elements and a .set of operations on them

Numbers and numerals 4.4

Numbers 4.4.1

Via the Arithmetic Is Construction metaphor, we conceptualize numbers as whole put together out of parts. The operations of arithmetic provide the patterns by which the part is arranged within the wholes. For example, every natural number can be .conceptualized uniquely as a product if prime numbers

Numerals 4.4.2

There is a big difference between numbers, which are concepts, and numerals, which are written symbols for numbers. In innate arithmetic, there are numbers but no numerals, since newborn children have not learned to symbolize numbers using numerals. The difference can be seen in Roman numerals versus Arabic numerals. Where the same number is represented by different numerals. For example, fourteen .is represented by XIV in Roman and by 14 in Arabic numerals with base ten

Calculation 4.4.3

Procedures for adding, subtracting, multiplying, and dividing less cognitive effort in positional notations than in non-positional notations like Roman numerals. Imaging !doing long division with Roman numerals

<u>Calculation Without Understanding</u> <u>4.4.4</u>

We can know how to use the algorithms for calculation without much understanding of what they mean. Algorithm, being freed from meaning and understanding, can be implemented in a physical machine called a computer, a machine that can calculate everything perfectly without understanding anything at all. When we learn procedure for adding, subtracting, multiplying, and dividing, we are learning algorithms for .manipulating symbols-numerals, not numbers

<u>:A comment</u>

אני לא מבין הוא איך פתאום המחברים מדברים על אלגוריתמים הריי דף קודם לכן הם דיברו על . newborn בפרק זה יש קפיצה/ פער מאוד גדול בין הפרקים הקודמים שדנים ביכולות של תינוקות לבין הפרק הנוכחי שדן באלגוריתמים וכדומה.

<u>Equivalent Result Frames and the Laws of Arithmetic</u> <u>4.5</u>

Familiarity with the various processes that achieve an identical result is an important part of our overall knowledge. From a cognitive perspective, such knowledge is represented in a conceptual frame within Charles Fillmore's theory of frame semantics :(1982, 1985). *An equivalent Result Frame* (ERF) include

,a desire result -
essential actions and entities, and -
a list of alternative ways of performing those actions with those entities to -
.achieve the result

For example, an important property of collections of objects can be state in terms of :the following ERF

<u>The Associative ERF for Collection</u>

:Desired result	A collection **N**
:Entities	Collections **A**, **B**, and **C**
:Operation	"Add to"
:Equivalent alternatives **A** added to (the collection resulting - from adding **B** to **C**) yields **N** the collection resulting from adding)- **A** to **B**) added to c yield **N**	

The metaphor Arithmetic Is Object Collection maps this ERF onto a corresponding :ERF for arithmetic

<u>The Associative ERF for Arithmetic</u>

:Desired result	A collection **N**
:Entities	Numbers **A**, **B**, and **C**
:Operation	"+"
:Equivalent alternatives **A**+(**B** + **C**)=**N**- **A**+ **B**)+**C**= **N**)-	

.This ERF expresses what we understand the associative law for arithmetic to mean Here we can see a clear example of how the grounding metaphors for arithmetic yield the basic laws of arithmetic, when applied to the ERFs for the source domain of .collections, construction, motion, and so on

Note, that the associative law does not hold for innate arithmetic, where all numbers must be subitizable-that is, less than 4. The reason, of course, is that closure does not hold for innate arithmetic. Thus, if we let **A**=1, **B**=2, and **C**=3, then **A+B+C=6.** Since 6 are beyond the usual range of what is subitizable, this assignment of these subutizable results to **A, B,** and **C** yield a result outside of innate arithmetic. This means that the associative law cannot arise in innate arithmetic and must arise elsewhere. As we have just seen, it arises in the source domain of the four grounding .metaphors

Why Calculation with Numerals Works 4.6

:The associative law works for the following reason

.The **4G**s ground our understanding and extend it from innate arithmetic-

The source domain of the **4G**s are object collection, object construction, physical - segmentation, and motion. Each of these is part of our understanding of the real .world

.The associative equivalent result frame is true of each physical source domain-

The **4G**s map those equivalent result frames onto the conceptual content of the - .associative law

The numeral-number mapping maps associative ERF for arithmetic onto the - .symbolized form of the associative laws: **A**+(**B** + **C**)= (**A**+**B**) + **C**

Because the symbolized equation corresponds to the cognitive content of the - equivalent result frame, the symbolic substitution yields an equivalent conceptual .result

Up to this point we have looked at the **4G**s and how they extend innate arithmetic and enrich it with properties like closure and the basic laws of arithmetic. We now .need flesh out the grounding metaphors discussed so far

Stretching the 4Gs 4.7

The **4G**s, as we have seen, are grounding metaphors that arise naturally from experience that conflates innate arithmetic with one of the domain. They also allow for extensions from natural numbers to other numbers. For example, the Arithmetic Is Motion Along a Path metaphor allow the path to be extended indefinitely from both sides of the origin, permitting zero and negative numbers to be conceptualized as point-locations on the path and therefore to be seen as numbers just like any other ".numbers. To achieve closure, the metaphors must be extended, or "stretched

Let us begin with addition and multiplication for negative numbers. Let negative numbers be point-locations on the path on the opposite the origin from positive numbers. The result in the source domain of the path is symmetry. For every point location at a given distance on one side of the origin, there is a unique point-location at the same distance on the other side. The symmetry point calls "**<u>symmetrical</u>** ".**<u>point</u>**
.Thus -5 is the symmetrical point of +5, and +5 is the symmetrical point of -5

<u>A comment :</u> Mental rotation is a natural cognitive operation. A 180 degree rotation around zero maps the positive numbers to the corresponding negative numbers, and vice versa. **<u>This cognitive operation provides grounding for metaphor for multiplication by negative numbers: Rotation by 180 degree Is Multiplication by .-1</u>**

<u>הערה:</u> Grounding metaphor = הכוונה היא שאלו המטאפורות <u>הבסיסיות</u> ביותר אשר מאפשרות לנו לפתח מושגים /כלים /חשיבות ...בתחומים שונים כגון מתמטיקה.

Since positive and negative are symmetric numbers, we need to distinguish them by picking an orientation: The usual choice is that positive numbers are on the right of .the origin and negative numbers are on the left
:The metaphorical mapping for addition, subtraction, and multiplication is as follow

Addition of positive numbers will now be conceptualized as moving toward the right, .whereas addition of negative numbers will be moving toward the left

Subtraction of positive numbers will be moving toward the left, while .subtraction of negative numbers will be moving toward the right

Multiplication by positive numbers require performing an action –moving-a certain number of times. Multiplication of a negative number –**B** by positive number **A** is: perform <u>repeated</u> addition of –**B** **A** times; that is starting from the Origin, and move **B** units to the left **A** times. **<u>But</u>** multiplication by a negative number is not a simple conceptual extension of multiplying by a positive number. A different metaphor is needed for multiplication by negative numbers. That metaphor must fit the laws of arithmetic. The symmetry between positive and negative numbers motivates a straightforward metaphor for multiplication by **–n.** first, do multiplication by the positive number **n** and then move (or "rotate" via a mental rotation) to the symmetrical point-the .point on the other side of the line at the same distance from the origin

For example, (-2)*5=-10, because 2*5=10 and the symmetrical point of 10 is -10. Similarly, (-2)*(-5)=10, because 2*(-5)=-10 and the symmetrical point of -10 is 10. Moreover, (-1)*(-1)=1 because 1*(-1)=-1 and the symmetrical point .of -1 is 1

The above process, from a cognitive perspective, is another metaphorical blend. Given the metaphor for multiplication by positive numbers, and given the metaphor for negative numbers and for addition, we form a blend in which we have both positive and negative numbers, addition for both, and multiplication for only positive numbers. To this conceptual blend we add the new metaphor for multiplication by negative numbers. To state the new :metaphor, we must use

,negative numbers as point-location to the left origin-
addition for positive and negative numbers in terms of movement, and-
multiplication by positive numbers in terms of repeated addition a positive-
.number of times, which results in a point-location

Only then we can formulate the new metaphor for multiplication by negative numbers .using concept of moving (or rotating) to the symmetrical point location

Multiplication by -1 Is Rotation

Source Domain Space	Target Domain Arithmetic
→ Rotation to the symmetry point of **n**	**n***1-

A comment: What is important to understand is the difference between the four basis grounding metaphors and extensions like the above one. The **4Gs** do arise naturally for natural activities correlated with subitizing and counting. They are what make the arithmetic of natural numbers natural. The laws of arithmetic for natural numbers are .entailments of those metaphors

Further Stretching 4.7.1

.It is easy to characterize division by negative numbers by stretching the **4Gs** .**(**See Page 93**)**

In short, the four basic grounding metaphors are natural and are constitutive of our fundamental understanding of arithmetic. That metaphor allows us to extend innate .arithmetic

What makes the arithmetic of natural number effective in the world are the four basic grounding metaphors uses to extend innate arithmetic and the metaphoric blend that .arise naturally from those metaphors

Summary

Let us review how arithmetic is "put together" cognitively. We can get an overall picture in terms of answers to a series of questions of the form "Where does X come :from cognitively?" where X includes all the following

.The conceptual numbers-
.The concept of closure-
.The laws of arithmetic-
.Fractions, zero, and negative numbers-
Generalization: Laws of arithmetic work for numbers in general, not just in - .specific cases
.Symbolization and calculation-
.The special properties of arithmetic-
.The fact that arithmetic "work" in so much of our experience-

In the rest of this chapter the authors answer the next question: "Where does .arithmetic as a whole come from?" The answer is a summary of the previous chapters

Methodology

.The various branches of cognitive science use a wide range of methodologies
In cognitive neuroscience and neuro-psychology, there are PET scan and fMRIs, the - study of the deficiencies of brain-damaged patients, animal experimentation, and so .on
In developmental psychology, there is, first of all, careful observation, but also - .experiments
In cognitive psychology, there is model building plus a relatively small number of - .experimental to gather data
In cognitive linguistics, the main technique is building models that generalize over - .the data

In this book the authors are building on results about innate mathematics from .neuroscience, cognitive psychology, and developmental psychology

Much of this book is concerned with conceptual metaphor. These metaphor studies mesh with studies showing that the conceptual system is **embodied** – that is, shaped .by the structure of our brain, our bodies, and everyday interaction in the world

The job in this chapter and throughout the book is to make the case that human mathematical reason work in roughly the same way as other forms of abstract reason- .that is, via sensory-motor grounding and metaphorical projection

:The constraints on the model-building methodology that used to account the data is

The grounding metaphors must be plausible, that is, they arise via conflation in - .(everyday experience (chapter 2
Given the sensory-motor source domain and the mapping, all the properties and - .computational inferences about the mathematical target domain must follow
In the case of arithmetic, the analysis must fit and extend what is known about innate- .arithmetic

.The model must be maximally general-
The model must accord with what is generally known about embodied cognition, that-.is, they must be able to fit with what is known about human brains and mind

A comment **The metaphor given so far are called grounding metaphors because they directly link a domain of sensory-motor experience to a mathematical domain**. But we shall see in the chapters to come, that abstract mathematics goes beyond direct grounding. The most basic forms of mathematics are directly grounded. Mathematics then uses other conceptual metaphors and conceptual blends to link one branch of mathematics to another. By means of **linking metaphors**, branches of mathematics that have direct grounding are extended to branches that have only indirect grounding. The more indirect the grounding in experience, the more "abstract" the mathematics is. **The mechanisms that linking abstract mathematics to that experiential grounding are conceptual metaphor and conceptual .blending**

In addition, a certain aspect of our linguistic capacities is used in mathematics-namely, the capacity for symbolizing. i.e. the capacity for associating written symbols with mathematical ideas. Mathematical symbols can be polysemous; that is they can have multiple, systematically associated meaning. For example 1 and 0 are sometimes used to mean true and false

.(The authors depended they research on the prior research of Ming Ming Chiu (1996

The next part of the book is the movement from basic arithmetic to more sophisticated .mathematics

Part II

Chapter 5

Essence and Algebra

By putting together what we know both from cognitive science and from history, we :can approach an answer to the next questions

?Why axioms are so important in mathematics-
Why it is so important for mathematicians to find the smallest number of -?independent axioms for a subject of matter
Why it was assumed for over two millennia (form Euclid until Godel) that a -whole mathematical subject matter should follow form a small number of postulates ?or axioms

The contribution from cognitive science is the notion of a **folk theory**-a cognitive structure characterizing a typically unconscious, informal "theory" about some .subject matter

A comment: "**Folk Psychology as a Theory**:" Many philosophers and cognitive scientists claim that our everyday or "folk" understanding of mental states constitutes a theory of mind. That theory is widely called "folk psychology" (sometimes "commonsense" psychology). The terms in which folk psychology is couched are the familiar ones of "belief" and "desire", "hunger", "pain" and so forth. According to many theorists, folk psychology plays a central role in our capacity to predict and explain the behavior of ourselves and others. However, the nature and status of folk .psychology remains controversial

The concept of "essence" is still with us, in the form of an unconscious folk theory about the world that most people in Western culture still take for granted. Here is that :folk theory

The Folk Theory of Essence 2.1

Every specific thing is a kind of thing. For example, there are kinds of animals: - lions, elephants, dogs… . The specific dog "Fido" is a kind of animal, in this case a .dog
Kinds are categories. All and only the members of the category have exactly that - .essence
Everything has an essence. For example, each elephant has certain features that - are essential to elephenthood: a trunk, large floppy ears, stumplike legs, and so on. .Those features constitute the "essence" of each elephant
Essences are causal. Essence and only essence determine the natural behavior of - things. For example, elephant eat the way they do because they have a trunk and because their legs cannot be used to pick up food. All natural behavior follows from .the essence. Moreover, the essence cannot follow from one another
.The essence of a thing is an inherent part of that thing-

:There are three basic metaphors for characterizing what an essence is. They are

.Essence Are Substances-
.Essence Are Forms -
.Essence Are Patterns of Changes -

The **folk theory** of essence is part of what constitutes our everyday "common sense" about physical objects; That is, it is part of the unconscious conceptual system that .govern our everyday reasoning

Euclid brought the folk theory of essence into mathematics in a big way. He claimed that only 5 postulates characterized the essence of plane geometry as a subject of matter. He believed that from this essence all other geometric truths could be derived by deduction-by reason alone! From this came the idea that every subject matter in mathematics could be characterized in terms of an essence-a short list of axioms, .taken as truths, from which all other truths about the subject matter could be deduced

Thus, the axiomatic method is the manifestation in Western mathematics of the folk theory of essences inherited from the Greek

Essence Within Mathematics 2.2

Numbers and other mathematical entities (e.g., triangles, groups, and topological spaces) are conceptualized as objects and therefore are assumed to have essences. It is the essence of a triangle to be a polygon with three angles. Axiom systems are taken .as defining the essence of each mathematical subject matter

Essence and Algebra 2.3

Algebra is about essence. It makes use of the same metaphor for essence that Plato .did-namely, Essence Is Form
Algebra is about abstract structure. It is that branch of mathematics that is conceptualized as characterizing essences in other branches of mathematics. In other word, mathematics implicitly assumes a particular metaphor for the essences of mathematical system: The Essence of a Mathematical System Is an Abstract .Algebraic Structure
Historically, algebra began with basic arithmetic structure (e.g., the operations of addition and multiplication over integers, the rational numbers, the real numbers, the complex numbers). Algebra has also come to include substructures of arithmetic (e.g., the numbers 0, 1, 2 with the operation of addition modulo 3). Algebra asks what the essence of each such structure is, where the essence of an arithmetic structure is taken :to include

,the elements in the structure-
the numbers and type of operations used on those elements, and-
the essential properties of the operations (i.e., the axioms governing the -
.(operations

Notice that this list does not include numbers specifically, it mentions only "elements." Nor does it include arithmetic operations like addition and multiplication; it mentions only "operation." In other words, algebra is "abstract;" it is about mathematical essence in general, across mathematical domains. Since the domain of mathematics from which algebra originally came is arithmetic, algebraic operations ".are often written "+" and "*" and are called "addition" and "multiplication

For example, the field that includes the elements {0, 1, 2} with addition modulo 3 (see page 111). The essence of this arithmetic structure is the set of elements, the binary operation of addition modulo 3, and the stated arithmetic laws governing over these elements, such as, Closure, Associative, Inverse, and so on. Algebra is not about particular essence but, rather, about *general essence*, considered as things in themselves. These general essences are called **"abstract"** –abstract elements, abstract laws, and abstract operations. For example, instead of the elements {1, 2, 3} we can take the abstract elements {I, A, B} and define the law and the operations on them, .(such as we get a field, (see page 112
What makes algebra a central discipline in mathematics is its relationship to other branches of mathematics, in which algebraic structures are conceptualized as the essence of other mathematical structure in other mathematical domain. This requires a

collection of what they call **Algebraic Essence Metaphors** (**AE metaphors** for short)-conceptual mapping from algebraic structure to structure in other mathematical .domains

For example to this metaphor see page 113. In short, the metaphor Algebraic Essence maps form the abstract algebra to specific algebra, that is, {I, A, B}→{0, 1, 2}, "*"→+, I→0, and so on. (Here "*" is a binary operation do not confuse with .(multiplication

A comment: The source domain of this metaphor is Algebra: Group, and the target domain is Modular Arithmetic. For example, in the source domain we have the abstract binary operation * that corresponded to the addition operation +, the algebraic-identity element I that corresponded to the arithmetic-identity element 0, the abstract associative law that corresponded to the arithmetic associative law, and .so on

Another metaphor is **Rotation Group Metaphor**. The source domain is Algebra: Group, and the target domain is Geometry Rotation. For example in the source domain we have the abstract elements {I, A, B}→{rotation in 120 degree, rotation in 240, rotation in 360}. The algebraic-identity element I→The rotational-identity element 360 degree rotation. The abstract element A→The 120 degree rotation, "*"→@ (where @ symbolizing two successive rotations), ("*" is an abstract operator in the source domain, and @ is an rotation operator in the target domain), and so on. .((See page 116

The two metaphors above are used by mathematicians to conceptualize addition modulo 3 and rotations of equilateral triangles as having the same mathematical .structure, *the same abstract essence*

Algebraic Essence Metaphors in General 2.4

When a mathematician claims that a particular mathematical system "has" or "form" a particular algebraic structure, that claim must substantiated via a complex mapping from the algebraic structure to the mathematical system. From a cognitive perspective, such a mapping is an AE metaphor that characterizes the essence of the mathematical system in terms of a structure in the domain of algebra. All AE :metaphor has the same form

The General Form of Algebraic Essence Metaphors

Source Domain Algebra	Target Domain A mathematical Domain **D** Outside of Algebra
→ An algebraic structure	The essence of a mathematical system in domain **D**
→ A set S_(A) of abstraction elements	A set S_(D) of elements of domain **D**
→ Abstract operations	Operations in domain **D**
→ (Algebraic identity element(s	Identity element(s) in domain **D**
→ (Other abstract element(s	Other element(s) in domain **D**

→ Abstract laws	Laws governing domain **D**
Tables for abstract operation(s) over the → (set S_(A	Table for operation(s) in domain **D** over (the set S_(D

:<u>A comment:</u> The table that mention in the last row of the above table is, for example

+Table

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Summary

The field of algebra begins with the Fundamental Metonymy of Algebra and builds on it, adding the metaphor Essence Are Forms-a metaphor inherited directly from Greek philosophy, which saw form as abstract and independent of substance. This is the source of the idea that abstract algebraic structures characterize the essential .forms of particular mathematical system

From the perspective of cognitive science, it begins with a metonymy and is based on a folk theory-the folk theory of essence-and a metaphor-The Essence of a Mathematical System Is an Abstract Algebraic Structure. It is this metaphor that links algebra to other branches of mathematics. We learn from this chapter what algebra *is*, :from a cognitive perspective

The folk theory of essence lies behind the very idea of abstract algebra. It tells us - how we normally think about algebra relative to other mathematical system. But the .folk theory of essence is not an account of mathematical cognition

The cognitive structure of an algebraic entity (e.g., a group) is not an essence that - inherent in other cognitive structures of mathematical entities (e.g., a collection of rotations). Rotation are conceptualized independently of groups, and groups are .conceptualized independently of rotations

Here is an example of how the folk theory of essence applies to algebra from a .cognitive perspective

Chapter 7

Sets and Hyper-sets

Intuitive sets are Container schemas, mental containers organizing objects into groups. For sets to be members in this conceptualization, they, too, must be conceptualized as objects. To conceptualize sets as objects, we need a conceptual **.**metaphor, hence this metaphor **Sets Are Objects**
This metaphor allow us to conceptualize *power sets,* in which all the subset of a **.(**given set **A** are made into members of another set, **P(A**

The Ordered Pair Metaphor 7.1

:The conceptualize of ordered pairs metaphorically in terms of sets is

The Ordered Pair Metaphor

Source Domain Sets	Target Domain Ordered Pairs
{{The Set {{a}, {a, b	(The Ordered Pair (a, b

Using this metaphorical concept of an ordered pair, one can go on to metaphorically .define relations, functions, and so on in terms of sets
Another use of this metaphor is to metaphorically conceptualize the natural numbers .(in term of sets that have other sets as members (John Von Neumann 1903-1957

The Natural Numbers Are Sets Metaphor

Source Domain Sets	Target Domain Natural Numbers
→ The empty set ϕ	Zero
The set containing the empty set {ϕ}(i.e., → ({{0	One
→ ({The set { ϕ , { ϕ }}(i.e., {0, 1	Two
The set { ϕ , { ϕ }, { ϕ , { ϕ }}}(i.e., {0, → ({1, 2	Three
→ And so on	A Natural Number

From a cognitive perspective, this is a metaphor that allows us to conceptualize .numbers, which are one kind of conceptual entity, in terms of sets
This is a **linking metaphor** – a metaphor that allows one to conceptualize one branch of mathematics (arithmetic) in term of another branch (set theory). Linking metaphors are different from grounding metaphors in that both the source and target domains of .the mapping are within mathematics itself

.**Q** I don't understand the different between the linking and the grounding metaphors

Cantor's Metaphor 7.2

Our ordinary conceptual system includes the concepts **Same Number As** and **More Than.** The are base on our experience with finite collections. For example, a group A has the same number of elements as group B if…. Group B has more objects than ….group A if

Cantor's Metaphor

Source Domain	Target Domain

Mappings	Numeration
Set **A** and set **B** can be put into one-to-one correspondence →	Set **A** and set **B** have the same number of .elements

For example, the next correspondence between the natural number and the even :number

$$2 \leftrightarrow 1$$
$$4 \leftrightarrow 2$$
$$6 \leftrightarrow 3$$
$$8 \leftrightarrow 4$$
$$10 \leftrightarrow 5$$
$$12 \leftrightarrow 6$$

.And so on

Axiomatic Set Theory and Hyper-sets 7.3

Axiomatic Set Theory 7.3.1

On the formalist view of the axiomatic method, a "set" is any mathematical structure that "satisfies" the axioms of set theory as written in symbols (The Zermelo-Fraenkel .axioms)**.** Many writers speak of sets as "containing" their members
The classic Zermelo-Fraenkel axioms, including the axiom of choise i.e., ZFC are presented in page 145. The axioms create the entity that called "set". The set used in this structure are not technically conceptualized as Container schemas. They do not have Container schema structure with an interior, boundary, and exterior. Sets are .undefined entities whose only constraints are that they must "fit" the axioms

Hyper-sets 7.3.2

Set theory has realized that a new non-container metaphor is needed for thinking about sets, and they have explicitly constructed one: hyper-set theory (Barwise & Moss, 1991). The idea is to use graph, not containers, for characterizing sets. The .(kinds of graphs used are Accessible Pointed Graphs, or APGs (see p.147
From the axiomatic perspective, hyper-set theorists have replaced the axiom of ZFC. From a cognitive point of view, the implicit conceptual metaphor they have used is :this

The Sets Are Graphs Metaphor

Source Domain Accessible Pointed Graphs	Target Domain Sets
→ An APG	The membership structure of a set
→ An arrow	The membership relation
→ Nodes that are tails of arrows	Sets
Decorations on nodes that are heads of arrow →	Members
→ APG with no loops	Classical sets with the foundation axiom
→ APG with or without loops	Hyper-sets with the Anti-Foundation

	axiom

The effect of this metaphor is to eliminate the notion of containment from the concept of a "set". Graph that have no loops satisfy the ZFC axioms. The axioms of "set theory" are not about what we ordinarily call "set", which we conceptualize in term of .containers

.An example for hyper-set present in page 149

:Until now we saw 3 kinds of metaphors
The grounding metaphors-metaphors that ground our understanding of :1 mathematical ideas of everyday experience. For example the 4 grounding metaphor .and Classes Are Container Schemas
Re-definitional metaphors-metaphors that impose a technical understanding :2 .replacing ordinary concept. For example, Cantor's metaphor
Linking metaphors-metaphors within mathematic itself that allow us to :3 conceptualize one mathematical domain in terms of another mathematical domain. For example, Von Neumann's Natural Numbers Are Sets metaphor, and Sets Are .Graphs metaphor

Part III
The Embodiment of Infinity

Chapter 8
The Basic Metaphor of Infinity

Infinity Embodied 8.1

To begin to see the embodied source of the idea of infinity, we must look to one of the most common of human conceptual systems, what linguists call the *aspectual system*. We conceptualize breathing, tapping, and moving as not having action .completions. This conceptualization is called *imperfective aspect*

A comment: Definition from Google: imperfective: aspect without regard to the) .(beginning or completion of the action of the verb

In chapter 2 we saw that the concept of aspect appears to be embodied in the motor-.control system of the brain
Given that the aspectual system is embodied in this way, we can see it as the fundamental source of the concept of infinity. Outside mathematics, a process is seen as infinite if it continues (or iterates) indefinitely without stopping. That is, it has imperfective aspect (it continues indefinitely) without an endpoint. This is the *literal concept of infinity* outside mathematics. It is used whenever one thinks of perpetual .motion-motion that goes on and on forever

Continuative Processes Are Iterative Processes 8.2

A process conceptualized and not having an end is called an imperfective process-one :that is not "perfect", that is, completed. Two subtypes of imperfective processes are

.(Continuative (Those that are continuous :1
Iterative (those that repeat and have an intermediate endpoint and intermediate :2 .(result

The idea of iterated action forms to express the idea of continuous action. This can be characterized in cognitive terms by the metaphor **Indefinite Continuous Processes .Are Iterative Processes**

Processes in general are conceptualized metaphorically in terms of motion; hence we conceptualizing indefinitely continuous motion as repeated motion. For example, continuous walking requires repeatedly taking steps, continuous swimming requires repeatedly moving the arms and legs. This conflation of continuous action and repeated actions gives rise to the metaphor by which continuous actions are .conceptualized in terms of repeated actions
This metaphor is used in the conceptualization of mathematics to break down continuous processes into infinitely iterating step-by-step processes, in which each .step is discrete and minimal

Actual infinity 8.3

Aristotle called to a motion without end or ongoing processes as *Potential infinity*. For .example, to writing down more and more decimals of √2
And he distinguished it from *actual infinity*. For example, infinite sets (like the set of natural numbers), points at infinity, limits of infinite series, infinite intersections and so on. All the cases of actual infinity, are special cases of a single general conceptual metaphor, they (the authors) called this metaphor as the **Basic Metaphor of Infinity** , or the **BMI** for short. The mechanism of this metaphor allows us to conceptualize the "result" of an infinite process that is in terms of a process that does have an end. The target domain of the BMI is the domain of processes without end-that is, what linguists call *imperfective processes.* The source domain of the BMI consists of an ordinary iterative process with an indefinite number of iterations with completion and .resultant state

The Basic Metaphor of Infinity

Source Domain Completed Iterative Processes	Target Domain Iterative Processes That Go On and On
→ The beginning state	The beginning state
State resulting from the initial stage of the → process	State resulting from the initial stage of the process
The process: Form a given intermediate → state, produce the next state	The process: Form a given intermediate state, produce the next state
The intermediate result after that iteration → of the process	The intermediate result after that iteration of the process
→ The final resultant state	**The final resultant state" (actual " (infinity**
Entailment E: The final resultant state is	**Entailment E: The final resultant state**

unique and follows every non-final →state	**is unique and follows every non-final state**

.**Q.** The table in p.159 needs explanation

They believe that *all* notion of infinity in mathematics can be seen as special cases of .the BMI

The Origin of the BMI Outside Mathematics 8.4

.The BMI is a general cognitive mechanism

.**Q.** The table in p.162 needs explanation

?What is Infinity 8.5

How do we, mere human being, conceptualize infinity? First, the answer must be biologically and cognitively plausible. That is, it must make use of normal cognitive and neural mechanisms. Second, the answer must cover all the cases in mathematics. Third, the answer must be sufficiently precise so that it can be demonstrated that these and other concepts of the infinite in mathematics can indeed be characterized as .special cases of a general cognitive mechanism
Their answer is that that general cognitive mechanism is the BMI. It has both neural and cognitive plausibility. In the next subsection they will show how it applies in a .wide variety of cases

∞ "The "Number 8.5.1

It is a mistake to think of infinity as a number-a mistake that many people make. If they (the authors) suggesting that a single cognitive mechanism-namely, the BMI-is used for all conceptions of infinity, then they have to show how the BMI can be used .to conceptualize infinity as a number, whether it is a "mistake" to do so or not

Cognitive sciences explain, as well, why people think as they do. The "mistake" of thinking of infinity as a number is not random. "∞" is usually used with a precise meaning-as a number in an enumeration, not as a number in a calculation. We see no cases of "17-∞*473. There are, cognitively, different uses for numbers-enumeration, comparison and calculation. As a number, ∞ is used in enumeration and comparison but not in calculation. The idea of ∞ as a number can also be seen as a special case of BMI. Note that the BMI does not nave any numbers in it. They apply the BMI to the .integers used to indicate order of enumeration

The BMI for Enumeration

Target Domain Iterative Processes That Go On and On	Special Cases The Unending Sequence of Integers Used For Enumeration
(The beginning state (0 →	No integers
State (1) resulting from the initial stage of the process →	The integer 1

The process: Form a prior intermediate state(n-1), produce the next state → ((n	Given integer n-1, form the next largest integer n
The intermediate result after that iteration of the process (the relation between n and → (n-1	n>n-1
The final resultant state" (actual " → (infinity	**∞ "The "integer**
Entailment E: The final resultant state is unique and follows every non-final → state	**Entailment E: The "integer" ∞unique and larger that every other integer**

The BMI itself has no numbers. However, the unending sequence of integers used for enumeration (but no calculation) can be a special case of the target domain of the .BMI

Projective Geometry: Where Parallel Lines Meet at Infinity 8.5.2

.In projective geometry, there is an axiom that *all parallel lines meet at infinity*

The Isosceles Triangle Frame

,An isosceles triangle ABC_n

:with sides AB, AC_n, BC_n

:Angles α_n, β_n

D_n : The distance from A to C_n

Where: $AC_n = BC_n, \alpha_n = \beta_n$

Inference: If AC_n lies on L_{1n} and BC_n lies on L_{2n}, then L_{1n} intersect L_{2n}

The iterative process in this case is to move point C_n further and further away from point A and B. As the distance D_n between A and C_n gets larger, the angles α_n and β_n approach 90 degree more and more closely. As a result, the intersecting lines L_{1n} and L_{2n} get closer and closer to being parallel. This is an unending, infinite process.
At each stage n, the lines meety at point C_n.
.(For more details, see Kline, 1962, ch. 11; Maor, 1987)

Here are the details for filling in the parameters in the BMI in this special case

Parallel Lines Meet at Infinity

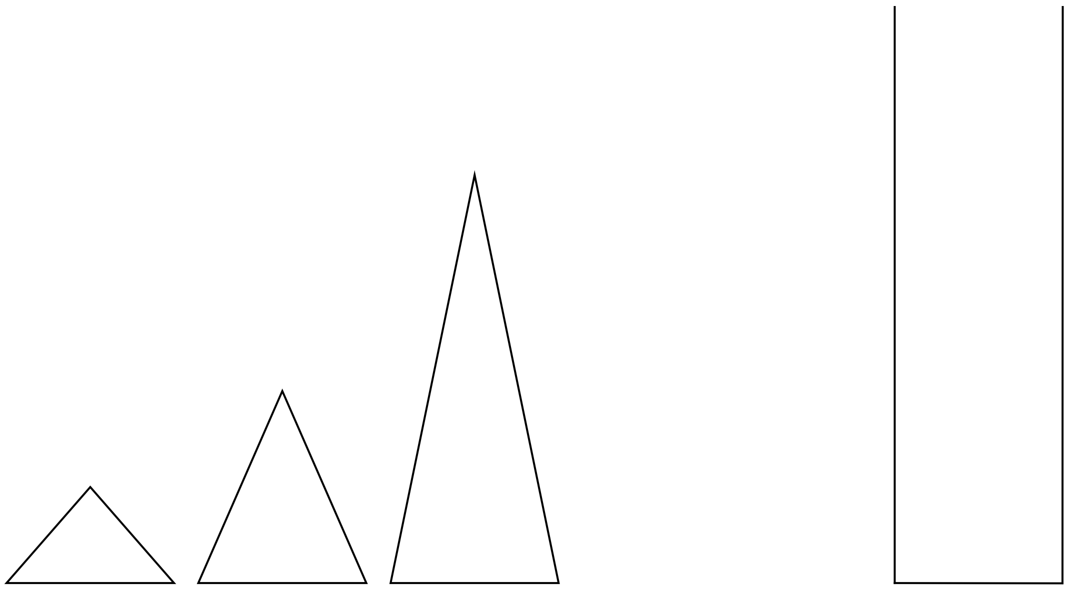
Target Domain Iterative Processes That Go On and On	Special Case Projective Geometry

→ (The beginning state (0	The isosceles - triangle frame, with triangle ABC_0
State (1) resulting from the → initial stage of the process	Triangle ABC_1, where the length of AC_1 is D_1
The process: Form a prior intermediate state(n-1), produce → (the next state (n	Form AC_n from AC_{n-1} by making D_n arbitrarily larger than D_{n-1}
The intermediate result after that iteration of the process (the → (relation between n and n-1	$D_n > D_{n-1}$ and $(90^0 - \alpha_n) < (90^0 - \alpha_{n-1})$
The final resultant state" " → ("∞" (actual infinity	$\alpha_\infty = 90^0$. Is infinitely long. Sides AC_∞ and BC_∞ are infinitely long, Parallel, and meet at C_∞ - a point "at infinity".
Entailment E: The final resultant state ("∞") is unique and follows every non-final state →	Entailment E : There is a unique AC_∞ (distance D_∞) that is longer than AC_n (distance D_n) for all finite n.

As a result of the BMI, lines and are parallel, meet at infinity,
and are separated by the length of line segment AB.

Figure 8.5.2. The application of the BMI to projective geometry. Drawing (a) show the isosceles triangle ABC_1, the first member of the BMI sequence.
Drawings (b) and (c) show isosceles triangle ABC_2 and ABC_3,
In which the equal sides get progressively longer and the equal angel get closer to 90^0.

:Drawing (d) shows the final resultant state of the BMI
ABC_∞
where the angle are 90 degree. The equal sides are infinitely long, and they ,
metaphorically "meet" at infinity-namely the unique point
C_∞



(a) (b) (c) … (d)

There is more examples in this chapter, like "The Point at Infinity in Inversive Geometry", "The Infinite Set of Natural Numbers", "Mathematical Induction", …"Generative Closure

Conclusions 8.6

The authors are discussing conceptual structure from a cognitive perspective, taking into account cognitive constrain, rather than from a purely mathematical point of view, which has no cognitive constraints whatsoever. That is, mathematicians are under no obligation to try to understand how mathematical understanding is embodied and how it makes use of normal cognitive mechanisms, like image schemas, aspectual .structure, conceptual metaphors, and so on

The hypothesis of the authors is testable (See Gibbs 1994). The hypothesis involves :certain sub-hypothesis

.The BMI is part of our unconscious conceptual system-
As with any cognitive model, it is hypothesized that the entailments of the model -
.are true of human cognition

Mathematics itself does not characterize what is common among the various concept of infinity. The common mathematical notion for infinity-"…", as in the sequence "1+1\2+1\4+…"-does not eve distinguish between potential and actual infinity. If it is potential infinity, the sum only gives an endless sequence of partial sums always less .than 2. If it is actual infinity, the sum is exactly 2

Q Why do we need to distinguish between the potential and actual infinity, the sum is !not equal to 2? It is not true mathematics

Their mathematical idea analysis, in this chapter, provides a clear and explicit answer to the question of how embodied being can have a concept of infinity-namely, via conceptual metaphor i.e., the BMI. And it shows the relationships among the various .ideas of infinity found in mathematics

Chapter 9

Real Numbers and Limits

The concept of infinity is central to the concept of real numbers and limits. In this chapter I will concentrate on the concept of limit. The ides of a limit arose in response to the fact that there are infinite series whose sums are finite. They will show how the BMI can conceptualize the mathematical ideas required to characterize limits of .infinite sequences

Limits of Infinite Sequences 9.1

A **limit** is a real number that the values of the sequence "get infinitely close to" as the number of terms increases "to infinity". We conceptualize the "convergence" as "approaching": The sequence "approach" the limit as the number of terms .""approaching infinity
The formal definition of a limit of a sequence does not capture the idea of :"approaching" a limit. The formal definition is

The sequence $\{x_n\}$ has L as a limit if, for each positive number ε,
there is a positive integer n_0 with the property that $|x_n - L| < \varepsilon$ for all $n \geq n_0$.

Their formulation uses the idea of the "process" in the BMI to characterize the process of "approaching" in the prototypical case. The process is one of getting "progressively closer", so that at each stage n the "distance" between the limit L and x_n becomes smaller. "Distance" in the geometric metaphor Number Are Points on a Line (see :chapter 12) is characterize metaphorically in terms of arithmetic difference
the absolute value $|x_n - L|$, which must become smaller as n becomes larger.
."That "distance" should "approach zero" as n "approach infinity

See figure 9.1 to understand how the BMI characterizes the idea of a sequence .approaching a limit

Here is what this looks like as a special case of the BMI. First we characterize the "Sequence and Limit frame", and then we use it in the special case of the BMI. Note that n is used to name the stages of the BMI and also to name the indexes of the terms of the sequence x_n

(The Sequence and Frame (Prototypical Version

A statement defining a sequence $\{x_n\}$.
A set S_n containing the first n terms of $\{x_n\}$.
A finite number L.
A set R_n of real numbers r such that $0 < r < |x_n - L|$.

(The BMI for Infinite Sequence (Prototypical Version

Target Domain Iterative Processes That Go On and On	Special Cases Infinite Sequence with a Limit L
→ (The beginning state (0	The Sequence and Limit frame
State (1) resulting from the → initial stage of the process	S_1 = The set containing the first .term of the sequence
The process: Form a prior intermediate state(n-1), produce → (the next state (n	From S_{n-1} containing the first n - 1 terms of the sequence, form S_n containing the first n terms of the sequence
The intermediate result after → that iteration of the	The set S_n. The set R_n containing all positive real numbers r such that $0 < r < \|x_n - L\|$. $R_n \subset R_{n-1}$
The final resultant state" " → ("∞" (actual infinity	The set S_∞ containing all the terms of the sequence. There is no positive real number r such that $0 < r < \|x_i - L\|$ for all x_i in S_∞. Hence, $R_\infty = \phi$. L is the limit of the sequence
Entailment E: The final	**Entailment E: L is the unique limit of the**

resultant state ("∞") is unique and follows every non- → final state	**.sequence**

The stages and the sets S_n at each stage

:is

<u>Stage n=1</u>	**<u>Stage n=2</u>**	**<u>Stage n=3</u>**	**<u>Stage n</u>**	**<u>∞=Stage n</u>**
$S_1 = \{x_1\}$	$S_2 = \{x_1, x_2\}$	$S_3 = \{x_1, x_2, x_3\}$	$S_n = \{x_1 x_2, ..., x_n\}$	$S_\infty = \{x_1 x_2, ...\}$

.A similar table could be built for the R_n 's

For example, Consider the sequence $\{x_n\} = \frac{n}{n+1}$

- The terms are: $x_1 = \frac{1}{2}, x_2 = \frac{2}{3}, x_3 = \frac{3}{4} x_4 = \frac{4}{5} ...$

- The sets $S_1 = \{\frac{1}{2}\}, S_2 = \{\frac{1}{2}, \frac{2}{3}\}, S_3 = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}\}, S_4 = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}\} ...$

- The limit $L = 1$

- The set $R_1 = \{r : 0 < r < \frac{1}{2}\}, R_2 = \{r : 0 < r < \frac{1}{3}\}, R_3 = \{r : 0 < r < \frac{1}{4}\} ...$

As n gets larger, the sets S_n come to have progressively more terms that get closer and closer to 1. The sets R_n come to exclude more and more positive real numbers. At stage n=∞, S_∞ contain all of the infinite number of terms of the sequence $\{x_n\} = \frac{n}{n+1}$.

It does not contain the number 1. At stage n=∞, there is no positive real number r such that $r < \left|\frac{n}{n+1} - 1\right|$ for every finit n. Thus, the set R_∞ is empty.

In this special case of theBMI, this is what it means for the infinite sequence $\frac{n}{n+1}$ to "approach" 1 as a limit as n "approaches ∞".

?**<u>Q</u>** How they get the limit 1

Answer: The value of the sequence is metaphorically conceptualizes as locations along the line. Visualizing the process via these metaphors, there are two coordinated trajectory in motion: As the first moves from integer to integer starting with 1, the second moves correspondingly from point-location to point-location on the number line, starting with 1/2. The first trajectory moves from 1 to 2, the second moves from 1/2 to 2/3. Via the BMI, ∞ is the endpoint of the line, the final point-location that the .first trajectory can metaphorically "approach" infinitely close to

When the first trajectory "reach infinity", the second trajectory "approach the limit"; that is, it gets to a sequence of point-location infinitely close to the limit, so close that there is no positive real number that can measure any distance between such point-.location and the limit

This is how the prototypical idea of "approach a limit as n approach infinity" can be .precisely conceptualized via the BMI and other metaphors

?**Q** I don't understand how the BMI metaphor computes the limit of the sequence

A comment The above case of the sequence is that the sequence converge directly toward the limit. But many sequences converge indirectly-winding around and going .back and forth as they ultimately converges to a limit

:This is the subject of the next subsection

The general Notion of a Limit Using the BMI 9.2

In this subsection the authors construct a fully general version of the BMI for .convergent sequence

Conclusion The authors shown in this chapter how a single cognitive mechanism-the BMI-with different special cases, can characterize the concept of actual infinity such .as: infinite decimals, limits of infinite sequence, infinite sum, and so on

Part V
Implication for the Philosophy Of Mathematics

Chapter 15
The Theory of Embodied Mathematics

:The basic questions that cognitive scientist asks is

?What are mathematical ideas from the perspective of cognitive science :1
?What commonplace cognitive mechanisms do they use
Given that innate mathematics is minuscule-consisting of subitizing and a tiny :2 bit of basic arithmetic-what cognitive mechanisms allow this tiny innate basic to be ?extended to generate all of advanced mathematics
?How are mathematical ideas grounded in our experience :3
?Which mathematical ideas are metaphorical and which are conceptual blends :4

Encounters with the Romance 15.1

The Romance of Mathematics

.Mathematics is an objective feature of the universe -
.Mathematical objects are real, mathematical truth is universal, absolute, and certain
What human being believes about mathematics there fore has no effect on what -
mathematics really is. Mathematics would be that same even if there were no human being, or being of any sort. Though mathematics is abstract and disembodied, it is .real
.Mathematical is the queen of the science-

(And so on (page 340

The romance provides the standard folk theory of what mathematics is for our .culture

Proof plays a central role in the romance. Proofs link human mathematicians to truths .of the universe. In the romance, proofs are discoveries of those truths
Is there any scientific evidence that this is true-that what is proved in human mathematics is an objective universal truth, truth on this physical universe or any possible universe, regardless of the existence of any being? **The answer is no. There is no such evidence!** (The argument is given in the Introduction of .(this book

?Is Mathematics in the physical World 15.2

Mathematics is used to describe phenomena in physical world and make correct predictions with great accuracy. "The universe runs according to mathematical laws", as if the laws came first and the physical universe "obeyed" the laws. Accordingly, the "truth" of physical laws formulated in mathematical terms is taken as indicating that the mathematics used in stating the physical laws is actually there in the physical universe. Since the regularities of the physical universe exist external to human being, so mathematics itself must exist .external to human being as part of the physical universe

The mathematics is in the minds of the physicists, not in the physical regularities themselves. It should be observed that most of the mathematics that has been done in the history of the discipline has no correlates at all. For Example, there are no parallel lines physically meet at infinity, no quaternion, no empty sets .and so on

Why the Only Mathematics Is Embodied Mathematics 15.3

The study of mathematical cognition may tell us about how we human being conceptualize and understand mathematics, how mathematics might be realize in the human mind and brain, how it might be learned, or how we make

mathematical discoveries, but it cannot tell us anything about mathematics !itself

<u>THE ONLY MATHEMATICS WE CAN KNOW IS THE MATHEMATICS .THAT OUR BODIES AND BRAINS ALLOWS US TO KNOW</u>

<u>Weak and Strong Requirements for a Theory of Embodied Mathematics</u> <u>15.4</u>

Everything we know either must be learned or must be built into the innate neural wiring of our brains. As we saw in Ch.1, it appears that the most fundamental aspect of arithmetic-the apprehension of small numbers of things-is indeed innate. But most of mathematical is not. The concept of zero, for example, is .not innate

When someone presents you with an idea, the appropriate brain mechanism must be in place for you to understand it and learn it. These are the *weak* requirements imposed on the cognitive science of mathematics by what we know about the embodiment of mind. But there are even stronger requirements (see Nunez, 1999). Cognitive science must explain how abstract reason is possible and how it is possible to have abstract concept and to understand them. The reason is that <u>abstract concept cannot be perceived by the senses</u>. We cannot see or hear or smell or touch the concept of justice, responsibility, and honor, much less the concept of ecological danger, evolution or entropy. It is challenge for cognitive science to explain-it terms of our bodies and brains-exactly how we can comprehend such concepts and think using them. The same is true of .every concept in advanced mathematics: limits, fractals, open sets and so on

The answer to the question: *How do we understand complex numbers ay all*? Cannot be a set of definitions, axioms, theorems, and proofs. That just pushes the question back one step further: What are the cognitive mechanisms involved in conceptualizing and learning those definitions, axioms, theorems, and ?proofs and all of the concept used in them

<u>The challenge for an Embodied Mathematics</u> <u>15.5</u>

Mathematics is *universal*, *precise, consistent* within each subject matter*, stable* over .time*, generalizable,* and d*iscoverable*

<u>The properties of Embodied Mathematics</u> <u>15.6</u>

:The theory of embodied mathematics makes the following claims

.Mathematics is a product of human being -

The parts of human cognition that generate advanced mathematics as an - enterprise are normal adult cognitive capacities-for example, the capacity for .conceptual metaphor

Simple numerations are built into human brains. For example, "subitize", this is - .clearly an embodied capacity

The subject matters of mathematics-arithmetic, geometry, calculus, set theory and - so on-arise from human concerns and activities. In other words, mathematics

is fundamentally a human enterprise arising from basic human activities. For .example, architecture, grouping, plying games and so on
.The mathematical aspect of these concerns is precision -
.Precision is greatly enhanced by the human capacity to symbolize -
Conceptual metaphor is a neurally embodied fundamental cognitive mechanism - that allows us to use the inferential structure of one domain to reason about another. It allows mathematicians to bring to one domain of mathematics the .ideas and the methods of precise calculation of another domain
Mathematical inferences and calculations tend not to change over time or spaces - .or culture
.Mathematics is not monolithic in its general subject matter -
Mathematics is effective in characterizing and making predictions about certain - .aspect of the real world as we experience it

The theory of embodied mathematics yield a new understanding of why mathematics is *universal*, *consistent* within each subject matter, *stable* in its inferences and calculations across people, culture, and time, able to *generalize* beyond humanly possible experience, *precise, symbolizable*, and *effective* for .describing major aspects of the natural world

The theory of embodied mathematics explicitly rejects any possible claim that mathematics is arbitrary shaped by history and culture alone. It is contrast to the form of postmodernism which claims that mathematics is purely .historically and culturally

Chapter 16

The Philosophy of Embodied Mathematics

The Embodied Nature of Mathematics

The cognitive science of mathematics has an immediate consequence for the philosophy of mathematics. It provide a new answer for perhaps the most :philosophical question of all regarding mathematics

:What I s the nature of mathematics? This answer can be summarized as follow

Mathematics, as we know it or can know it, exists by virtue of the embodied - .mind
.All mathematical content resides in embodied mathematical ideas -
A large number of the most basic, as well as the most sophisticated, mathematical - .ideas are metaphorical in nature

Ontology and Truth 16.1

:Two of the central questions of the philosophy of mathematics have been

?What are mathematical objects -
?What is mathematical truth -

:Possible answers

.Mathematical objects are real, objectively existent entities -
.Mathematical truth is objective truths of the universe -

But such answers are ruled by the cognitive science of mathematics in general and :mathematical ideas analysis in particular. So there is a new answer

Mathematical object are embodied concept-that is, ideas that ultimately grounded - in human experience and put together via normal human conceptual mechanisms, .such as image schemas, conceptual metaphors, and conceptual blends

.Mathematical truth is like any other truth -

Mathematical entities can only be conceptual in nature, arising from our embodied conceptual system. A mathematical statement can be true only if the way we understand that statement fits the way we understand the subject matter that the .statement is about. Conceptual metaphors often enter into those understanding

Some example 16.2

:To make these very abstract views clearer, here some examples

Example 1 16.2.1

Entity: Zero Truth: n+0=n

.Zero is not a subitized number and it is not a part of innate mathematics
As we saw in chapter 3 and 4, zero has become a number through the metaphorical :extension of the natural numbers. Thus

.In the **Arithmetic Is Object Collection** metaphor: zero is the empty collection
.In the **Arithmetic Is Object Construction** metaphor: zero is the lack of an object
.In the **Measuring Stick** metaphor: zero is the initial point of the measurement
.In the **Arithmetic Is motion** metaphor: zero is the origin of motion

.It is entailment of each of these metaphors that $n+0=n$

In the **object-collection** metaphor, adding a collection of *n* objects to an empty .collection yields a collection with n objects

In the **moving** metaphor, taking *n* steps from the origin and then taking no steps .leaves you at the same place as just taking *n* steps from the origin

Given our understanding of zero and our understanding of the operation addition and .its result, we can see why we must take $n+0=n$ to be true

Example 2 16.2.2

: Entity: The infinite sum : Truth

This infinite sum "exist" by virtue of the **Basic Metaphor of Infinity**. The truth is "true" by virtue of the **Basic Metaphor of Infinity** (see chapter 8 and 9). The existence of this entity and this truth depends upon the acceptance of the idea of actual (as opposed to potential) infinity. If one rejects such a notion, as many .mathematicians have, then there is no such entity and no such truth

Without such conceptual metaphor, those entities do not exist and those truths do not hold. For example, zero does not exist conceptually without the grounding metaphors .for arithmetic, and the "truth" $n+0=n$ is not "true" if there is no zero

The Point 16.3

An entity existing conceptually as a result of a metaphorical idea. In each case, the truth is "true" only relative to that metaphorical idea. Each such mathematical entity exist only in the minds of being with those metaphorical ideas, and each mathematical .truth is true only in the minds of being with such mathematical ideas

Formalist mathematics is a remarkable intellectual program that rests upon a fundamental metaphor, what they (the authors) will call the *Formal Reduction .Metaphor*

The Formal Reduction Metaphor 16.4

This conceptual metaphor provides a prescription for

Conceptualizing any mathematical subject matter in terms of sets and :1
Symbolizing any such subject matter in a uniform way that meshes with formal :2 logic. Accordingly, what are called mathematical "proofs" can be symbolized and .made mechanical

The target domain of this metaphor consists of mathematical concepts in general. The source domain consists only of sets, structures within set theory, symbols, and string .of symbols

The metaphor provides a way to project a set-theoretical structure onto any domain of mathematics at all. According to this metaphor, any conceptual structure in any .branch of mathematics can be mapped onto a set-theoretical structure

.(The map of this metaphor and an example are in p.369)

The Formal Reduction Metaphor is actually a metaphorical schema, since it generalizes over all branches of mathematics. Each particular reduction for a branch .of mathematics would have a somewhat different special case of the schema

The Formal Reduction Metaphor

Source Domain Sets and Symbols	Target Domain Mathematical Ideas
A set-theoretical entity (e.g., a set, a → (member, a set-theoretical structure	A mathematical concept
→ An ordered n-tuple	An n-place relation among mathematical concepts
→ A set of ordered pairs	A function or operator
→Constrains on a set theoretical structure	Conceptual axioms: ideas characterizing the essence of the subject matter
Inherently meaningless symbol strings → combined under certain rules	The symbolization of ideas in the mathematical subject matter
Inherently meaningless symbol string → "called "axioms	The symbolization of the conceptual axioms-the ideas characterizing the essence of subject matter
A mapping (called an "interpretation") from the inherently meaningless symbol → string to the set-theoretical structure	The symbolization relation between .symbols and the ideas they symbolize

.A comments: 1: This metaphor conceptual metaphor
When we learn a formal mathematics, we learn how to apply the :2
.Formal Reduction Metaphor

What is Equality 16.5

No more than a few hundred years old mathematicians used instead of "=" was language to express what they meant. For example, they used they words: "yield", "gives", "produces", "can be decompose into", "can be factored into", "results in", and so on. For instance, "3+2=5" is usually understood to mean that the operation of adding 3 to 2 yields 5 as a result. Here 5 is the result of a process of addition and "=" .establishes the relationship between the process and the result

But, "5=3+2" is usually understood as the number 5 can be decomposed into the sum .of 3 plus 2

From a cognitive perspective, there is no single meaning of "=" that covers all these cases. An understanding of what "=" means requires a cognitive analysis of the mathematical ideas involved. Moreover, none of these meanings of "=" is abstract and disembodied. Every such meaning is ultimately embodied via our everyday .experience plus the metaphors and blends we use extend that experience

A Portrait of Mathematics 16.6

;Here is the view of mathematics that emerges from this book

Mathematics is a natural part of being human. It arises from our bodies, our - .brains, and our everyday experiences in the world
There is nothing mysterious, mystical, and magical about mathematics. It is a - consequence of human evolutionary history, neurobiology, cognitive capacities, and .culture
.Mathematics is one of the greatest products of the collective human imagination -
Mathematics is a system of human concepts that makes extraordinary use of the - .ordinary tools of human cognition
The effectiveness of mathematics in the world is a tribute to evolution and to - .culture
.Everything in mathematics is comprehensible-at least in principle -
We have learned from the study of the mind that human intelligence is - multifaceted and that many forms of intelligence are vital to human culture. .Mathematical intelligence is one of them
.Mathematics is creative and open-ended -
Human conceptual systems are not monolithic. They allow alternative version of - concepts and multiple metaphorical perspectives of many important aspects of our .lives
Mathematics is a magnificent example of the beauty, richness, complexity, - .diversity, and important of human ideas
Human being has been responsible for the creation of mathematics, and we - .remain responsible for maintaining and extending it

Part VI

$$e^{\pi i}+1=0$$

A Case Study of the Cognitive Structure Of Classical Mathematics

This equation of Leonhard Euler's is one of the deepest in classical mathematics. It :brings together in one equation the most important numbers in mathematics

$$e, \pi, i, 1, and\ 0.$$

This one equation implicitly states a relationship among the branches of classical mathematics-arithmetic, algebra, geometry, analytic geometry, trigonometry, calculus, .and complex variables

:Some questions are that arise from this equation are

What could it mean to multiply a real number, *e*, by itself an imaginary number - .of times i.e. by *i* or by π times
?What could "√-1 times" possible mean -
?Why should $e^\pi i$ equal, of all things, -1 -
.$e^\pi i$ has an imaginary number in it, but the result -1 is real number -

Case Study 1
Analytic Geometry and Trigonometry

?Exactly what does e^πi mean and why does it equal to -1

Knowing how to prove something does not necessarily mean that we understand the .deep meaning of what we have proved

To answer for the above questions we must look closely at the cognitive mechanisms that connect different branches of mathematics. Not surprisingly, **linking metaphor** .will be involved

Case Study1.1 **The Functions Are Numbers Metaphor**

We conceptualize functions as ordered pairs of points in the Cartesian plane. The operations of arithmetic can be metaphorically extended from numbers to functions, so that functions can be metaphorically added, subtracted, multiplied, and divided in a :way that consistent with arithmetic. Here is the metaphor

Functions Are Numbers

Source Domain Numbers (Value of Functions)	Target Domain Functions
Any arithmetic operation on the y-values → (of two functions y=f(x) and y=g(x	The corresponding arithmetic operation (on the two functions f(x) and g(x

For example, the sum (f+g)(x) of two functions f(x) and g(x) is the sum of the value of those functions at x:f(x)+g(x). Note that the "+" in "f(x)+g(x)" is the ordinary literal "+" that is used in the addition of two numbers, while the "+" in "f+g" is metaphorical, a product of this metaphor. Similarly, all the following are special cases :of this metaphor

(f-g)(x) Is f(x)-g(x) -
(f*g)(x) Is f(x)*g(x) -
.f/g)(x) Is f(x)/g(x), where g(x) not 0) -

Case Study 1.2 **The Basic Metaphor of Trigonometry**

The fundamental metaphor that defines the field of trigonometry conceptualizes .angle as numbers. This metaphor called by the authors **Trigonometry Metaphor**

The Unit Circle blend has 4 corresponding domains, and it will build up in stages. .((see page 388 for the stages

The Unit Circle Blend

Domain 1 A Circle in the Euclidian Plane with Center and Radius	Domain 2 The Cartesian Plane with X-axis, Y-axis, (Origin at (0,0	Domain 3 An Angle Φ in the Euclidian Plane with Leg 1 and Leg 2	Domain 4 A Right Triangle, with Hypotenuse, Right Angle, <A, <P, Side a and Side b
Euclidian plane Center	Cartesian plane Origin	Euclidian plane Vertex of Φ	Euclidian plane Vertex of <A
Radius	Distance 1 from origin	-------------	Length of hypotenuse
-------------	The interval [0,1] on the X-axis	Leg 1	-------------
Line from center to point P on the circle	Line from origin to point (x_1, y_1)	Leg 2	Hypotenuse

Arc subtended	Arc length	Size of Φ	Size of <A
---------------	A segment from $(0,0)$ *to* $(x_1,0)$	--------------	Side a
---------------	A segment from (x_1, y_1) *to* $(x_1,0)$	--------------	Side b

.(The explanation of the table and the 4 stages is in pages 388-192 includes figures)

<u>Case Study 1.3</u> **<u>The Trigonometry Metaphor</u>**

:Given the Unit Circle blend, it is now possible to state the Trigonometry metaphor

<u>The Trigonometry Metaphor</u>

Source Domain The Unit Circle Blend	Target Domain Trigonometric Functions
The length of the arc subtended by angle → Φ	The number assigned to angle Φ
→ The length of side a	(The value of the function cos(Φ
→ The length of side b	(The value of the function sin(Φ

Entailment

→ The unit circle: x^2+y^2=1	$\sin^2(\Phi)+\cos^2(\Phi)=1$
→ Φ is a right angle	$\Phi=\frac{\pi}{2}$
→ Φ spans half of the circle	$\Phi=\pi$
→ Φ spans the whole circle	$\Phi=2\pi$
→ Φ is zero	$\sin(0)=(\textit{which is equals}\,\sin(\pi))=0$
→ (Φ is two right angles (180 degree	$\cos(\pi)=-1$

<u>A comment:</u> Note that the <u>length</u> of the arc is the number assigned to angle Φ. Since the diameter of the unit circle is 2, its circumference is 2π. Thus, if Φ spans the whole

circle, the number assigned to it by the Trigonometry metaphor will be 2π. If Φ spans .half the circle, it follows that the number assigned to it will be π

Case Study 1.4 **The Polar Coordinate Metaphor**

Given the Trigonometry metaphor, we can now form a further metaphorical blend :from that metaphor

The Trigonometry Blend

Source Domain The Unit Circle Blend	**Target Domain** **Trigonometric Functions**
The length of the arc subtended by angle → Φ	The number assigned to angle Φ
→ The length of side a	(The function cos(Φ
→ The length of side b	(The function sin(Φ

This allows us to use the Trigonometry blend to conceptualize points in the Cartesian .plane using sin and cosine functions. The results are polar coordinates

The Polar-Trigonometry Metaphor

Source Domain The Trigonometry Blend	Target Domain The Cartesian Plane Blend
→ r	The distance from the origin O to point (P=(a, b
→ θ	The angle formed by a counter clockwise (from the X-axis to OP, where P=(a, b
(The ordered pair (r, θ →	(The point (a, b
→ (rcos(θ	a
→ (rsin(θ	b

.**Q** I don't understand the source domain of this metaphor

Each point in the Cartesian plane (a, b) is uniquely characterized by the ordered pair of numbers (r, θ), where r is a number (the distance of (a, b) from the origin), and θ is .a number by virtue of the Trigonometry metaphor

What makes polar coordinate metaphorical? Recall that the Cartesian plan in itself has no trigonometry-no angles-as-numbers, no sin and cos. Through this metaphor, the Cartesian plane becomes conceptualized in terms of trigonometry, with each point (a, b) being conceptualized as the pair of numbers (r, θ). As we shall see, this .metaphor is crucial in understanding *e^πi+1=0*

This metaphor **link** together arithmetic, algebra, and geometry to create two whole .new branches of mathematics-analytic geometry and trigonometry

.The meaning of π conceptual: a ratio of circumference to diameter

Case Study 2
?What is e

The expression *e^ πi,* the number *e*=2.718281828459045… raised to the power *π*i*=3.1415926…..*√-1, cannot means *e* multiplied by itself *π* times and the result .then multiplied by itself √-1 number of times

The explanation of this problem is the **exponential function (or the log function)** as mapping from the domain of real numbers under addition to the range of positive real .numbers under multiplication

From a cognitive perspective, the exponential function can be seen as a conceptual metaphor-namely, **Multiplication Is Addition** -where products are conceptualized in terms of sums and calculated by addition. The exponential function is an .arithmetization of this metaphor

.For more details see p.399-406

Case Study 2.1 **What is e**

."First see the metaphor in page 406 **"The Mathematical Metaphor for Change** :In chapter 14 they discussed on the concept of derivative

$$\frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

:Now suppose that

$$f(x) = b^x.$$

:Than

$$\frac{d}{dx} b^x = \lim_{\Delta x \to 0} \frac{b^{x+\Delta x} - b^x}{\Delta x}.$$

Because the exponential function is a function mapping form sums to product, the :following equality holds

$$\frac{d}{dx}b^x = \lim_{\Delta x \to 0}\frac{b^{x+\Delta x} - b^x}{\Delta x} = \lim_{\Delta x \to 0}\frac{b^x b^{\Delta x} - b^x}{\Delta x} = \lim_{\Delta x \to 0}\frac{b^x(b^{\Delta x} - 1)}{\Delta x}$$

Since b^x contains no Δx term, it can be taken outside the limit sign, and because the :limit exists, let symbol it by C, than

$$\frac{d}{dx}b^x = C \cdot b^x.$$

In other words, the derivative of an exponential function with base b-namely, b^x-is constant times b^x itself, which is another way of saying that _exponential function .has a rate of change proportional to its size_

When C=1, than the rate of change of b^x is b^x itself, that is, b^x has a rate of change that is identical to its size. In mathematical terms, b^x is its own derivative. :This happen when

$$\lim_{\Delta x \to 0}\frac{b^{\Delta x} - 1}{\Delta x} = 1$$

.That limit constrains the value of the base b
e" is the name they give to the base b when that limit is 1. It means that *e^x* is a " .function whose rate of change is identical to itself

:Since exponential functions map sums onto product, we can now see what *e* means

<u>The Concept of *e*</u>

e" is that number which is the base of a function"
That maps sums onto products and
.Whose rate of change is identical to itself

:The numerical value of *e* is determined by the following limit

$$\lim_{\Delta x \to 0}\frac{e^{\Delta x} - 1}{\Delta x} = 1$$

.....Euler found a way to calculate this limit. The answer is 2.71828128459045

.(There are two answers in the book for this number see page 411-415)

:For example, one answer is the given by Power-Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

:Setting x=1 in the power series, we get

$$e^x = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + ...$$

.....The value of this series converges to 2.71828128459045

What is the Function

e^x

:It is the function that

,maps sums onto products -

,maps 2.718281828459045…onto 1 -

is its own derivative, and -

.changes in exact proportion to its size-

What is *e*

….*e* is the real number 2.718281828459045 -

e is the base of an exponential function that has -

.a rate of change exactly equal to its size

Thus, *e* is a real number that is the base of a function that maps sums onto products -

.and whose rate of change is exactly that same as its size

Case Study 3

?What Is *i*

להשלים מהמקומי

This blend cerats a new conceptual entity-the rotation plane, which is the Cartesian plane together with the metaphor characterizing multiplication by -1 as a rotation by .180 degree

Since the Rotation-Plane Blend has the Euclidean plane within it, a relevant theorem :of the Euclidean plane holds

.Each pair of intersecting line segments uniquely determines a parallelogram

.(See p.427)

.The rotation plane has the following arithmetic properties

Arithmetic Properties of the Rotation Plane

.*(a,b)+(c,d)=(a+c,b+d)*
.*(c,0)(a,b)=(ca,cb)*
.*(a,b)=(-a,-b)(1,0-)*
.is the additive identity *(0,0)*
.is the multiplicative identity *(1,0)*
.*(*The additive inverse of *(a,b)* is *(-a,-b*
Addition and scalar multiplication are associative and commutative
.Scalar multiplication is distributive over addition

Q How they inferred the law of multiplication: *(c,0)(a,b)=(ca,cb)* or *(-1,0)(a,b)=(-a,-* .*(b)*. (p.426

The answer is in page 426: Given a point (a,b) in the Cartesian plne, ri\otation by 180 degree take the point to the point (-a,-b). In the Rotation-Plane Blend, this means (-1,0)(a,b)=(-a,-b). But why -1 in the Rotation-Plane Blend is (-1,0)? Maybe the answer :for this question is the follow from the correspondence in the table in p.426

The number line ⇔ Every line through the origin

Case study 3.3 **The 90^0 Rotation Plane**

Recall that the rotation plane is essentially just the Cartesian plane to which we have added the metaphor characterizing multiplication by -1 as a rotation by 180 .degree

(כלומר מישור הסיבוב הוא ביסודו המישור הקרטזי שהוסיפו לו את המטאפורה המאופינת ע"י הכפלה ב 1- כסיבוב ב 180 מעלות).

In the rotation plane, rotation by 180 degree is multiplication by a number. But what rotaion by 90 degree has no arithmetic correlate. Hence

The 90 degree Rotation Metaphor

Source Domain Space	
Rotation by 90^0 →	.Multiplication by *n*

Two rotation by 90^0 $\rightarrow$	Multiplication by n^2

They call the result of adding this metaphor to the rotation plane the *90 degree rotation plane.* In the 90 degree rotation plane, all the arithmetic properties of the Rotation-Plane Blend will be preserved. But the additional possibilities added by the .90 degree Rotation metaphor will yield new properties

What is the product of the number *n* and *(0,1)*? A rotation by 90 degree applied to :*(1,0)* is *(0,1)*. Thus

.(n(1,0)=(0,1

????**Q** Why

:What is the product of *n* and *(a,b)*? A rotation of *(a,b)* by 90 degree yield *(-b,a)*. Thus

.(n(a,b)=(-b,a

:In the same way

$$n^2(a,b) = (-a,-b).$$

Since (-1,0) in the rotation plane is just -1 on the Number Line, embedding the :number line in the 90 degree rotation plane yield the result that

$$n^2 = -1.$$

(Since

$$n^2(1,0) = (-1,0).$$

.(

And

$$n = \sqrt{-1}.$$

Which is what has been called "*i*". Renaming *n* as *i=√-1* gives us the following version of the 90 degree Rotation metaphor, when it occurs as part of the rotation .plane

<u>(The 90 degree Rotation Metaphor (As Part of the Rotation Plane</u>

Source Domain Space	Target Domain Arithmetic
→ Rotation by 90 degree	Multiplication by i $(= \sqrt{-1})$
→ Two rotation by 90 degree	Multiplication by i^2 $(= -1)$

?Case study 3.3.1 **What Is the Complex Plane**

As note, the rotation plane is just the Cartesian plane with the structure imposed by our normal metaphor for conceptualizing multiplication by -1 namely, Multiplication .by -1 Is Rotation to the Symmetry Point on the Line

The **complex plane** is just the 90 degree rotation plane-the rotation plane with the structure imposed by the 90 degree Rotation metaphor added to it. Multiplication by I .is "just" rotation by 90 degree